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STABILITY DIAGRAMS OF THE PERIODIC MOTIONS OF A PENDULUM WITH AN OSCILLATING AXIS*

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Periodic rotations of a pendulum with a harmonically oscillating axis of suspension are studied analytically and numerically. General regularities in their bifurcation diagrams are established, depending on the evenness of the numbers characterizing the number of rotations of the pendulum and the period of oscillations of the axis of suspension.

The phenomenon of the dynamic stability of the upper position of the pendulum and the effect of vibrational excitation and of the maintenance of its rotations have found wide application in modern devices and mechanisms /1-3/. The mathematical models of the motions of a parametrically excited pendulum in the form of non-linear, non-autonomous differential equations, taking resistance forces into account, were studied by analytical methods and a number of periodic modes were investigated numerically (see /2/ where a survey of the investigations and a bibliography are given, and also /4-6/). In the Hamiltonian case the periodic motions of a rotational body have not been studied before.

The present paper deals with periodic rotations of a parametrically excited non-linear oscillator, without taking the dissipation into account. The Cesari method is used to obtain the generating solutions, a number of periodic rotations of a single type are established and their stability is studied in the case when the values of two parameters are small. A number of solutions of practical interest are continued numerically into the domain of large values of the parameters. The bifurcation diagrams

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of periodic solutions are constructed, containing domains of their stability (to a first approximation) and the bifurcations of the appearance and change in the stability of these solutions are clarified.

1. Formulation of the problem. The simplest mathematical model of the motions of a pendulum whose axis of support executes harmonic oscillations along the vertical, is represented by the equation /7/

$$x^{\cdot\cdot} + (a + b\cos 2t)\sin x = 0 \tag{1.1}$$

where x is the deflection of the pendulum from the vertical, and a and b are parameters. The problem consists of determining the conditions for the occurrence and stability of periodic motions of the pendulum synchronous with the oscillations of its axis, i.e. of solutions $\Gamma_{p,q}$ of Eq.(1.1) which will satisfy the condition $x(t + p\pi) = x(t) + 2\pi q$ where $p = 1, 2, \ldots$, $q = 0, \pm 1, \pm 2, \ldots$ Here $|q|$ denotes the number of rotations of the pendulum about the axis in the positive direction $(q > 0)$ or negative direction $(q < 0)$ over p periods of the oscillations of the axis, and $q = 0$ corresponds to states of rest or to oscillatory motions about the vertical.

2. Periodic solutions for small values of the parameters. When $a = b = 0$, Eq.(1.1) has an infinite set of solutions

$$x(t) = 2tr + \alpha, \quad r = q/p, \quad \alpha = \text{const} \in [0, 2\pi) \tag{2.1}$$

where r is a simple fraction belonging to the interval $(-\infty, \infty)$. Let us find the values of α corresponding to the generating solutions and elucidate the nature of the stability of the periodic solutions $\Gamma_{p,q}$ for small values of a and b.

We shall use the Cesari method* (*Sarychev V.A. and Sazonov V.V., On a method of investigating the periodic solutions of ordinary differential equations. Preprint 105, Moscow, In-t Prikl. Matematiki, 1976.) to obtain the values of α as simple roots of the bifurcation equation $V(\alpha, a, b) = 0$. The stability of the generating solutions $\Gamma_{p,q}$ is determined by the sign of the derivative $dV/d\alpha$ calculated at the corresponding α. When the condition $dV/d\alpha \leqslant 0$ holds, the solution $\Gamma_{p,q}$ is stable (to a first approximation). If the inequality does not hold, then $\Gamma_{p,q}$ is unstable.

We will make the substitution $u = x - 2tr$ and write $a = a_m b^m$ where m is a positive integer and a_m is a constant. We obtain the equation

$$u^{\cdot\cdot} + (a_m b^m + b\cos 2t)\sin(u + 2tr) \tag{2.2}$$

Following the Cesari method we shall construct an auxiliary system

$$u^{\cdot\cdot} + (a_m b^m + b\cos 2t)\sin(u + 2tr) - V(\alpha, b) = 0, \quad \int_0^{p\pi} u\, dt = \alpha \tag{2.3}$$

The solution of this system consists of two functions $u = u(t, \alpha, b)$, $V = V(\alpha, b)$ represented by the series

$$u(t, \alpha, b) = \alpha + \sum_{k=1}^{\infty} u_k(t, \alpha) b^k, \quad V(\alpha, b) = \sum_{k=1}^{\infty} v_k(\alpha) b^k \tag{2.4}$$

where $u_k(t, \alpha)$ are the $p\pi$-periodic functions of the variable t. The presence of a simple root $\alpha(b)$ of the bifurcation equation $V(\alpha, b) = 0$ represents the necessary and sufficient condition for the existence of a unique, $p\pi$-periodic solution of Eq.(2.2) transforming, as $b \to 0$, into the solution $u \to \alpha(0)$. If $v_i(\alpha) \equiv 0\ (i = 1, 2, \ldots, k - 1)$, $v_k(\alpha) \not\equiv 0$ in (2.4) and a value of α_l exists for which $v_k(\alpha_l) = 0$, $dv_k(\alpha_l)/d_\alpha \not\equiv 0$, then the bifurcation equation has a simple analytic root $\alpha(b) \to \alpha_l$ as $b \to 0$. This means that in order to determine the roots of the equation $V(\alpha, b) = 0$ for small $b > 0$ it is sufficient to find the first coefficient $v_k(\alpha) \not\equiv 0$ and investigate its roots.

With this purpose in mind, we shall substitute series (2.4) into system (2.3), expand the function $\sin(u + 2tr)$ in a Taylor series and equate the coefficients of like powers in b. This yields the system

$$u_k^{\cdot\cdot} = F_{k-1}(t, \alpha)\cos 2t + F_{k-m}(t, \alpha) a_m - v_k(\alpha) \tag{2.5}$$

$$\int_0^{p\pi} u_k\, dt = 0 \quad (m = 1, 2, \ldots; k = 1, 2, \ldots) \tag{2.6}$$

where

$$F(t,\alpha)=\sum_{s_j\in U_n}[i\sigma_n\cos(\alpha+2tr)+(\sigma_n-1)\sin(\alpha+2tr)]\times$$
$$\prod_{j=0}^{n}\frac{(iu_j)^{s_j}}{s_j!},\quad n\geqslant 0$$
$$F(t,\alpha)=0,\quad n<0;\quad u_0=\alpha,\quad \sigma_n={}^1/_2[1-(-1)^{\lambda}],\quad \lambda=\sum_{j=0}^{n}s_j$$

U_n is the set of collections of integers j and s_j satisfying the conditions

$$s_0=0,\quad s_j\geqslant 0,\quad \sum_{j=0}^{n}js_j=n$$

The system (2.5), (2.6) contains, apart from the unknown functions $u_k(t,\alpha), v_k(\alpha)$, the unknown constants a_m and m. In the case of a simple fraction r, we shall determine these values by successive determination of the solution of system (2.5), (2.6) for $m, k = 1, 2, \ldots$, beginning with the conditions for the existence of a non-trivial function $v_k(\alpha)$. For $m = 1$, $k = 1$ Eq. (2.5) will be written in the form

$$u_1^{\cdot\cdot} = -(a_1+\cos 2t)\sin(\alpha+2tr)-v_1(\alpha) \tag{2.7}$$

Integrating this equation we obtain a solution depending on two arbitrary constants C_1 and C_2. Since the functions $u_k(t,\alpha)$ must be $p\pi$-periodic in t, we shall assume, from now on, that $C_1 = C_2 = 0$. Then the solution of (2.7) will become (here and henceforth the upper and power plus and minus signs will be in correspondence)

$$u_1 = -{}^1/_2(a_1\sin\alpha+v_1)t^2+{}^1/_4\sin\alpha\cos 2t,\quad r=0 \tag{2.8}$$
$$u_1 = {}^1/_4 a_1\sin(\alpha\pm 2t)+{}^1/_{32}\sin(\alpha\pm 4t)-{}^1/_2(v_1+{}^1/_2\sin\alpha)t^2,\quad r=\pm 1 \tag{2.9}$$
$$u_1=\left(\frac{1}{2r}\right)^2 a_1\sin(\alpha+2tr)+\frac{\sin[\alpha+2t(r+1)]}{8(r+1)^2}+\frac{\sin[\alpha+2t(r-1)]}{8(r-1)^2}-\frac{1}{2}v_1t^2,\quad r\neq 0,\pm 1 \tag{2.10}$$

For $r = \pm 1$ the condition (2.6) determines the coefficient $v_1(\alpha) = -{}^1/_2\sin\alpha$. The equation $v_1(\alpha) = 0$ has simple roots $\alpha_1 = 0$ and $\alpha_2 = \pi$ which corresponds to two generating solutions with $p = q = 1$ and two solutions with $p = 1, q = -1$. The function $v_1(\alpha)$ does not depend on a_1, and this means that for any value of a and b two solutions $\Gamma^i_{1,1}$ and two solutions $\Gamma^i_{1,-1}$ $(i = 1, 2)$ exist. When a, b are sufficiently small, the condition $dV/da \leqslant 0$ can be replaced by the inequality

$$dv_k(\alpha_i)/d_\alpha \leqslant 0 \tag{2.11}$$

When checking (2.11) it was found that the stable solutions $\Gamma^1_{1,1}$ and $\Gamma^1_{1,-1}$ correspond to the root $\alpha_1 = 0$, and unstable solutions $\Gamma^2_{1,1}$ and $\Gamma^2_{1,-1}$ to the root $\alpha_2 = \pi$.

When $q = 0$, we find that $v_1(\alpha) = -a_1\sin\alpha$. The roots $\alpha_1 = 0$ and $\alpha_2 = \pi$ of the equation $v_1(\alpha) = 0$ correspond to the stationary solutions $x = 0$ and $x = \pi$, and we shall denote them by $\Gamma^1_{1,0}$ and $\Gamma^2_{1,0}$. From (2.11) it follows that the solution $\Gamma^1_{1,0}$ $(\Gamma^2_{1,0})$ is stable when $a_1 > 0$ $(a_1 < 0)$, and unstable when $a_1 < 0$ $(a_1 > 0)$. We find that the value $a = a_1 = 0$ corresponds to the change in the stability of the stationary solutions. Additional investigation of this bifurcation is carried out below at $m = 2$.

When $r \neq 0, \pm 1$, condition (2.6) yields $v_1(\alpha) \equiv 0$. Therefore we write $k = 2$ in (2.5) and consider the equation

$$u_2^{\cdot\cdot} = -(a_1+\cos 2t)\cos(\alpha+2tr)u_1-v_2(\alpha) \tag{2.12}$$

Substituting (2.10) into (2.12) we obtain

$$u_2=\left[\frac{a_1^2}{2^7r^4}+\frac{1/r^2+1}{2^6(r^2-1)^2}\right]\sin 2(\alpha+2tr)+\frac{a_1}{2^6(2r-1)^2}\times\left[\frac{1}{(r-1)^2}+\frac{1}{r^2}\right]\sin 2(\alpha+t(2r-1))+\frac{a_1}{2^6(2r+1)^2}\left[\frac{1}{(r+1)^2}+\frac{1}{r^2}\right]\sin 2(\alpha+t(2r+1))+\frac{\sin 2(\alpha+2t(r-1))}{2^9(r-1)^4}+\frac{\sin 2(\alpha+2t(r+1))}{2^9(r+1)^4}-\frac{a_1r\sin 2t}{2^4(r+1)^2(r-1)^2}-\frac{r\sin 4t}{2^7(r-1)^3(r+1)^3}-v_2(\alpha),\quad r\neq 0,\pm{}^1/_2,\pm 1 \tag{2.13}$$

$$u_2 = \frac{1}{8}\left(a_1^2 + \frac{5}{18}\right)\sin 2(\alpha \pm t) \mp \frac{a_1}{18}\sin 2t + \frac{1}{32}\sin 2(\alpha \pm t) + \frac{1}{2592}\sin 2(\alpha \pm 3t) \mp \frac{1}{144}\sin 4t - \frac{a_1}{4}t^2 \sin 2\alpha + \frac{a_1}{576}\sin 2(\alpha \pm 2t) - \frac{v_2(\alpha)}{2}t^2, \quad r = \pm\frac{1}{2}. \tag{2.14}$$

In the case of $r = \pm{}^1/_2$, condition (2.6) determines $v_2(\alpha) = -{}^1/_2 a_1 \sin 2\alpha$. The equation $v_2(\alpha) = 0$ has the roots $0, \pi/2, \pi, 3\pi/2$. The values $\alpha = 0$ and $\alpha = \pi$ correspond to the generating solution $x_1 = \pm t$, $\alpha = \pi/2$, and $\alpha = 3\pi/2$ to the solution $x_2 = \pm t + \pi/2$. As in the case if $q = 0$, the solution $\Gamma^1_{2,\pm 1}$ is stable when $a_1 > 0$ while $\Gamma^2_{2,\pm 1}$ is unstable, and their stability changes when $a_1 = a = 0$.

For $r \neq 0, \pm{}^1/_2, \pm 1$ the coefficient $v_2(\alpha) = 0$ for any a_1. When $k = 3$, Eq. (2.5) takes the form

$$u_3^{\cdot\cdot} = -(a_1 + \cos 2t)[u_2 \cos(\alpha + 2tr) - {}^1/_2 u_1^2 \sin(\alpha + 2tr)] - v_3(\alpha) \tag{2.15}$$

Substituting (2.10) and (2.13) into (2.15) and integrating, we obtain the coefficient $u_3(t, \alpha)$ which is lengthy and is therefore omitted. Using (2.6) we find for $r = \pm{}^1/_3$

$$v_3(\alpha) = -\frac{3}{100}\left(\frac{7}{16} + \frac{281}{25}a_1^2\right)\sin 3\alpha.$$

We find that we have, for any a_1, six roots $\alpha_s = \pi s/3$ $(s = 0, 1, \ldots, 5)$. The roots α_s with even s (including $s = 0$) correspond to the generating solution $x_1(t) = \pm{}^2/_3 t$, and those with odd s to the solution $x_2(t) = \pm{}^2/_3 t + {}^1/_3\pi$. A check of the condition (2.11) showed that the first solution generates a stable solution $\Gamma^2_{3,\pm 1}$, and the second an unstable solution $\Gamma^2_{3,\pm 1}$. Using (2.6) for $r = \pm{}^3/_1$ we obtain $v_3(\alpha) = -\sin\alpha$. The equation $v_3(\alpha) = 0$ has simple roots $\alpha_1 = 0$ and $\alpha_2 = \pi$. The root α_1 corresponds to the stable solution $\Gamma^1_{1,\pm 3}$ and α_2 corresponds to the unstable solution $\Gamma^2_{1,\pm 3}$.

For $r = \pm{}^2/_3$ and $r = \pm{}^2/_1$ we have $v_3(\alpha) = -a_1 \sin 3\alpha$ and $v_3(\alpha) = -a_1 \sin\alpha$. When studying the roots of the equation $v_3(\alpha) = 0$, we found that the generating solutions are $x_1 = \pm{}^4/_3 t$, $x_2 = \pm{}^4/_3 t + {}^1/_3\pi$ and $x_1 = \pm 4t$, $x_2 = \pm 4t + \pi$. As in the case of $r = 0, \pm{}^1/_2$, a change in the stability of the generated solutions $\Gamma^1_{3,\pm 2}, \Gamma^2_{3,\pm 2}, \Gamma^1_{1,\pm 2}, \Gamma^2_{1,\pm 2}$ occurs when $a = 0$. For $r \neq \pm{}^1/_3$, ${}^3/_1$, $\pm{}^2/_3$, $\pm{}^2/_1$ the function $v_3(\alpha) \equiv 0$. Continuing this process for $k = 4, 5, \ldots$, we can determine the value and character of the stability of the solutions $\Gamma_{p,q}$ for other values of r.

The results of solving systems (2.5), (2.6) for $m = 1$ and integral values of k ranging from 1 to 10 have shown that in the case of odd values of p and q we can find the value of the fraction q/p determined by the system (2.5), (2.6) with odd value of k. To do this we must construct the sequences of all irreducible fractions $\{s/k\}$ where s is an odd number, $0 < s/k \leqslant 1$. Then every fraction s/k will have a corresponding pair of fractions $p/|q|$ $(p = s, q = \pm k)$ and $|q|/p$ $(q = \pm s, p = k)$. Here the equation $v_k(\alpha) = \gamma_{p,q}\sin p\alpha = 0$ $(\gamma_{p,q} < 0)$ with $2p$ roots $\alpha_l = \pi l/p$ $(l = 0, 1, \ldots, 2p - 1)$ corresponds to the fraction q/p. The roots with an even index (including $l = 0$) correspond to the stable solution $\Gamma^1_{p,q}$, and those with an odd index to the unstable solution $\Gamma^2_{p,q}$. This enables us to formulate the following assertion.

Theorem 1. When a and b are sufficiently small, Eq. (1.1) will have, for any pair of prime numbers p and q, stable (to a first approximation) and unstable periodic solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ generated by the solutions $x_1(t) = 2tq/p$ and $x_2(t) = (2tq + \pi)/p$ respectively.

Let us explain the bifurcation which occurs when the nature of the stability of the solution $\Gamma_{p,q}$ change when $a = 0$ for the prime numbers p and q, one of which is even. As in the case of $m = 1$, we have found for $m = 2, 3, \ldots, 8$ the functions $u_k(t, \alpha)$ and $v_k(\alpha)$ for all k from 1 to 14.

Let us first formulate some general conclusions which follow from the results obtained, and then illustrate them with specific examples.

1°. The coefficient $v_k(\alpha) \equiv 0$ for all values of $k \neq 2(m-1), 2m$ (this includes the case in which the inspection of the roots of the equation $v_k(\alpha) = 0$ does not yield any new data as compared with $m = 1$).

2°. If $k = 2(m-1), 2m$, then the function $v_k(\alpha)$ has the form

$$v_k(\alpha) = -\sin p\alpha\,(a_m A_k + B_k \cos p\alpha) \tag{2.16}$$

where A_k and B_k are positive constants depending on p and q.

3°. In order to establish the values of p and q which are determined by the system (2.5), (2.6) with $k = 2(m-1)$ and $k = 2m$, we must construct the corresponding sequences $\{s/(m-1)\}\,(0 \leqslant s/(m-1) < 1)$ and $\{s/m\}\,(0 < s/m < 1)$ of all irreducible fractions where s is an even (odd) number, $s = 0$ when $m = 2$ (for the first sequence). The fraction $|q|/p$ corresponds to every fraction $s/(m-1)$, and the fraction $p/|q|$ to s/m.

As the first example we shall study the solutions $\Gamma_{1,0}$ corresponding to the lower or upper position of equilibrium of a pendulum, or to its oscillations about them. When $m = 2$ and $k = 1$, we obtain Eq. (2.7) not containing a_1. Condition (2.6) determines $v_1(\alpha) \equiv 0$ for the solution (2.8), which confirms the statement 1°. When $k = 2(m-1) = 2$, we have

$$u_2'' = -a_2 \sin\alpha - u_1 \cos 2t \cos\alpha - v_2(\alpha)$$

Let us substitute (2.8) into the above expression and find the solution. Using (2.6) we obtain the function $v_2(\alpha) = -\sin\alpha\,(a_2 + {}^1/_8 \cos\alpha)$. Apart from the roots $\alpha_1 = 0$ and $\alpha_2 = \pi$ (compare $m = 1$), the equation $v_2(\alpha) = 0$ has, at $|a_2| = |a/b^2| < {}^1/_8$, simple roots $\alpha_3 = \arccos(-8a/b^2)$, $\alpha_4 = 2\pi - \arccos(-8a/b^2)$ which correspond to π-periodic solutions $\Gamma^3_{1,0}$ and $\Gamma^4_{1,0}$ of the oscillatory type. The roots α_3 and α_4 are continuous functions of the parameters a and b. When the value of a/b^2 increases within the interval $(-{}^1/_8, {}^1/_8)$, the root α_3 increases from zero to π and α_4 decreases from 2π and π. Since the function $v_2(\alpha)$ is 2π-periodic, it follows that when $a = -b^2/8$, the roots α_3 and α_4 merge with the root $\alpha_1 = 0$, while when $a = b^2/8$, they merge with $\alpha_2 = \pi$.

We find that apart from two stationary solutions $\Gamma^1_{1,0}: x(t) = 0$ and $\Gamma^2_{1,0}: x(t) = \pi$, two periodic solutions $\Gamma^3_{1,0}$, $\Gamma^4_{1,0}$ exist in the region $|a/b^2| < {}^1/_8$ for small a and b. The periodic solutions of the oscillatory type merge, on the curves $l_{1,0}$: $a = -b^2/8$ and $l'_{1,0}$: $a = b^2/8$ with the solutions $\Gamma^1_{1,0}$ and $\Gamma^2_{1,0}$ respectively.

In order to clarify the nature of the stability of the solutions, we shall obtain the derivative $dv_2/d\alpha = -((a/b^2)\cos\alpha + 1/8 \cos 2\alpha)$ and calculate its value at $\alpha = \alpha_i$. Using (2.11) we find that the solution $\Gamma^1_{1,0}$ is stable in the region $a > b^2/8$, and $\Gamma^2_{1,0}$ is unstable. Conversely, in the region $a < -b^2/8$ the solution $\Gamma^1_{1,0}$ is unstable and $\Gamma^2_{1,0}$ is stable. When $|a| < b^2/8$ both solutions are stable and periodic solutions $\Gamma^3_{1,0}$ and $\Gamma^4_{1,0}$ corresponding to the roots α_3 and α_4 are unstable. Thus the change in the nature of the stability of the solutions $x(t) = 0$ and $x(t) = \pi$ on the curves $l_{1,0}$ and $l'_{1,0}$ respectively is accompanied by the appearance of two unstable periodic solutions $\Gamma^3_{1,0}$ and $\Gamma^4_{1,0}$. This agrees with numerical results obtained for the stationary and periodic solutions $\Gamma_{1,0}$.* (*Batalova Z.S. and Belyakova G.V., On the structure of phase space of the equations of motion of a pendulum with an oscillating point of support. Gor'kii, 1984. Dep. in VINITI, 31.05.84: 3539-84.)

Further we take, for $m = 2$, the value $k = 2m = 4$. The sequence $\{s/m\}$ contains a single fraction ${}^1/_2$, i.e. we obtain for the case $q/p = \pm{}^2/_1$ the non-trivial function $v_4(\alpha)$. Without dwelling on the solution of system (2.5), (2.6) with the values $k = 1, 2, 3$ for which the function $v_k(\alpha) \equiv 0$, we shall write the Eq. (2.5) at $k = 4$

$$u_4'' = -a_2[u_2 \cos(\alpha \pm 4t) - {}^1/_2 u_1^2 \sin(\alpha \pm 4t) - \cos 2t[(u_3 - {}^1/_6 u_1^3) \times \cos(\alpha \pm 4t) - u_1 u_2 \sin(\alpha \pm 4t)] - v_4(\alpha)$$

Substituting here the expressions found for u_1, u_2, u_3, we obtain the solution $u_4(t, \alpha)$. The condition (2.6) determines the coefficient $v_4(\alpha) = -\sin\alpha\,(288 a_2 + 25\cos\alpha)$ and from this it follows that the equation $v_4(\alpha) = 0$ has, for any a_2, simple roots $\alpha_1 = 0$ and $\alpha_2 = \pi$. Moreover, when $|a_2| = |a/b^2|$, roots $\alpha_3 = \arccos(-a/(\varkappa b^2))$, $\alpha_4 = 2\pi - \alpha_3$ ($\varkappa = {}^{25}/_{288}$) exist. When a/b^2 increases from $-\varkappa$ to $\varkappa$, the root α_3 increases from zero to π and α_4 decreases from 2π to π.

Using these data and condition (2.11) we find that for sufficiently small a and b there exist two bifurcation curves $l_{1,2}: a = -\varkappa b^2$ and $l'_{1,2}: a = \varkappa b^2$ which divide the neighbourhood of the point $(0, 0)$ into three regions. In the region $|a| < \varkappa b^2$ four solutions exist $\Gamma^i_{1,\pm 2}$ $(i = 1, 2, 3, 4)$ where $\Gamma^1_{1,\pm 2}$ and $\Gamma^2_{1,\pm 2}$ are stable to a first approximation, while $\Gamma^3_{1,\pm 2}$ and $\Gamma^4_{1,\pm 2}$ are unstable. In the region $a > \varkappa b^2$ $(a < -\varkappa b^2)$ we have a stable solution $\Gamma^1_{1,\pm 2}$ $(\Gamma^2_{1,\pm 2})$ and an unstable solution $\Gamma^2_{1,\pm 2}$ $(\Gamma^1_{1,\pm 2})$. The change in the stability of the solutions $\Gamma^1_{1,\pm 2}$ and $\Gamma^2_{1,\pm 2}$ occurs on the bifurcation curves $l_{1,2}$ and $l'_{1,2}$ respectively, leading to the appearance of unstable solutions $\Gamma^3_{1,\pm 2}$ and $\Gamma^4_{1,\pm 2}$.

In the next example we take $m = 3$. We shall find the values of p and q for which the roots of the bifurcation equation can be found. In the case of $k = 2(m-1) = 4$ and $k = 2m = 6$ the sequences shown in Sect.3 contain the functions ${}^1/_2$ and ${}^2/_3$ respectively; therefore the consecutive solution of system (2.5), (2.6) yields a non-trivial function $v_4(\alpha)$ for $q/p = \pm{}^1/_2$ and the function $v_6(\alpha)$ for $q/p = \pm{}^3/_2$.

In the case of $q/p = \pm 1/2$ the equation $v_4(\alpha) = -1/4 \sin 2\alpha\,(3a_3 + 1/8 \cos 2\alpha) = 0$ has, for any a_3, simple roots $0, \pi/2, \pi, 3\pi/2$. The roots 0 and π correspond to the generating solution $x_1(t) = \pm t$ and the roots $\pi/2$ and $3\pi/2$ to the solution $x_2(t) = \pm t + \pi/2$. When $|a_3| = |a/b^2| < 1/24$, the equation also has the roots $\alpha_1^3 = 1/2 \arccos\,(-a/(24b^3))$, $\alpha_2^3 = \pi + \alpha_1^3$, $\alpha_1^4 = 2\pi - \alpha_1^3$, $\alpha_2^4 = \pi - \alpha_1^3$, which correspond to two generating solutions $x_3(t) = \pm t + \alpha_1^3$ and $x_4(t) = \pm t + \alpha_1^4$. When a/b^3 increases in the interval $(-1/24, 1/24)$, the roots α_1^3 and α_2^3 increase from **zero** to $\pi/2$ and from π to $3\pi/2$ respectively, while the roots α_1^4 and α_2^4 decrease from 2π to $3\pi/2$ and from π to $\pi/2$. Taking these data and condition (2.11) into account, we obtain the following result for the analysis of the stability of the solutions $\Gamma^{i}_{2,\pm 1}$ generated by the solutions $x_i(t)$. For sufficiently small a and b we have two bifurcation curves $l_{1,1}: a = -b^3/24$ and $l_{1,2}: a = b^3/24$, which divide the neighbourhood of the point $(0, 0)$ into three regions. When $|a| < b^3/24$, we have two stable solutions $\Gamma^1_{2,\pm 1}$ and $\Gamma^2_{2,\pm 1}$, and two unstable solutions $\Gamma^3_{2,\pm 1}$ and $\Gamma^4_{2,\pm 1}$. In the region $a > b^3/24$ the solution $\Gamma^2_{2,\pm 1}$ is stable and $\Gamma^1_{2,\pm 1}$ is unstable, while in the region $a < -b^3/24$ the converse is true, namely $\Gamma^1_{2,\pm 1}$ is unstable and $\Gamma^2_{2,\pm 1}$ is stable. When the point (a, b) passes across the bifurcation curve $l_{2,1}$ ($l'_{2,1}$) into the region $|a| < b^3/24$, the change in stability of the solution $\Gamma^1_{2,\pm 1}$ ($\Gamma^2_{2,\pm 1}$) is accompanied by the appearance of solutions $\Gamma^3_{2,\pm 1}$ and $\Gamma^4_{2,\pm 1}$.

In the last example, where $m = 7$, we shall show the values of p and q for which the values of the roots of the bifurcation equation have been found. When $k = 2(m-1) = 12$, the sequence of irreducible fractions (see 3°) contains the fractions $1/6$, $5/6$. The non-trivial function $v_{12}(\alpha)$ of the form (2.16) with $p = 6$ is obtained for the values $q/p = \pm 1/6$ and $q/p = \pm 5/6$. When $k = 2m = 14$, the sequence contains the fractions $3/7$, $5/7$, $6/7$. The function $v_{14}(\alpha)$ with $p = 2, 4, 6$ is obtained for $q/p = \pm 7/2, \pm 7/4, \pm 7/6$.

In all these cases we have found two bifurcation curves $l_{p,q}$ and $l'_{p,q}$ on which the stability of periodic solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ changes and two unstable solutions $\Gamma^3_{p,q}$ and $\Gamma^4_{p,q}$ appear. The results of the special cases of $a = 0$ and $q = 0$* (* Batalova Z.S. and Bukhalova N.V., Periodic oscillations of a pendulum with oscillating vertical axis of rotation. Gor'kii, 1984, Depd. in VINITI 01.02.84, 618-84. See also the previous footnote.) discussed earlier lead to the following general assertion.

Theorem 2. For sufficiently small a and b there exist, for any pair of prime numbers p and q, one of which is even (including $q = 0$), periodic solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ generated by the solutions $x_1(t) = 2tq/p$ and $x_2(t) = (2tq + \pi)/p$ respectively. The bifurcation curves $l_{p,q}: a = -a_m b^m$ and $l'_{p,q}: a = a_m b^m$ $(a_m > 0)$ emerging from the point $(0,0)$, divide the neighbourhood of this point into three regions. To the right of the curve $l'_{p,q}$ the solution $\Gamma^1_{p,q}$ is stable and $\Gamma^2_{p,q}$ is unstable, to the left of the curve $l_{p,q}$ the solution $\Gamma^1_{p,p}$ is unstable and $\Gamma^2_{p,q}$ stable. A change in the stability of the solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ occurs on the curves $l_{p,q}$ and $l'_{p,q}$ respectively, and is accompanied by the appearance of another two solutions $\Gamma^3_{p,q}$ and $\Gamma^4_{p,q}$. Four solutions exist in the region situated between the curves $l_{p,q}$ and $l'_{p,q}$, namely the stable $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ and unstable $\Gamma^3_{p,q}$ and $\Gamma^4_{p,q}$.

3. Domains of existence and stability of periodic rotations of the pendulum.

A set of algorithms and programs /8/ was used to extend the investigation of periodic rotations $\Gamma_{p,q}$ to the domain of large values of the parameters a and b. The coordinates of the initial point $N_{p,q}(x(0), x^{\cdot}(0))$ corresponding to the solution $\Gamma_{p,q}$ were determined in the bounded domain $G_{p,q}$ of variation of the parameters, and the multiplying factors ρ_1 and ρ_2 characterizing its stability in the first approximation were found. The step Δ in the parameters a and b was chosen depending on the magnitude of the multiplying factors. For the values of $|\rho_{1,2}|$ close to +1 the value of Δ did not exceed 10^{-3}, and for the other values of $\rho_{1,2}$ the solution was traced using the step $\Delta \leqslant 0{,}2$. The results of investigating the solutions $\Gamma_{p,q}$ were given in the form of stability diagrams in the a, b-parameter plane and of the graphs showing the dependence of the initial data $x(0)$ and $x^{\cdot}(0)$ on a and b. The graphs were given for the solution $\Gamma_{p,q}$ and $q > 0$, since for the equal values of a and b the solutions with $q > 0$ and $q < 0$ have the same character of stability and the corresponding initial points are situated symmetrically about the x-axis (see the reference in time second footnote above).

Let us now consider specific periodic solutions $\Gamma_{p,q}$ of Eq.(1.1). The form of some of the solutions is shown in Fig.1.

The solutions $\Gamma_{1,1}^1$ and $\Gamma_{1,1}^2$ generated by the solutions $x(t) = 2t$ and $x(t) = 2t + \pi$ were studied in the domain $G_{1,1}\{|a|\leqslant 2, 0\leqslant b\leqslant 7\}$. From Sect.2 it follows that when $a=b=0$, the solutions have the corresponding initial points $N_{1,1}^1(0,2), N_{1,1}^2(\pi,2)$. When a and b increase, the values of the abscissas of these points do not change, and remain equal to zero, and π. The graphs showing the dependence of the ordinates $x^{\cdot}(0)$ are given for the solutions $\Gamma_{1,1}^1$ and $\Gamma_{1,1}^2$ (the solid and dashed line) in Fig.2a.

Calculating the multiplying factors ρ_1 and ρ_2 of the solution $\Gamma_{1,1}^2$ shows that when a and b increase from zero, ρ_1 increases from unity and $\rho_2=\rho_1^{-1}$, i.e. the solution $\Gamma_{1,1}^2$ is unstable in the region $G_{1,1}$. Analysis of the multiplying factors of the solution $\Gamma_{1,1}^1$ enabled us to construct the bifurcation diagram shown in Fig.2b. The solution $\Gamma_{1,1}^1$ is unstable within the shaded regions, the vertical shading corresponds to negative values of $\rho_{1,2}$ and $|\rho_1|<1, |\rho_2|>1$, and oblique shading to positive values. Non-shaded regions correspond to the domains of stability of the solution $\Gamma_{1,1}^1$; here $\rho_{1,2}=\exp(\pm i\varphi(a, b))$. At the points on the curves $l_{1,1}$ and $d_{1,1}$ we have $\rho_{1,2}=-1$ and at the points $s_{1,1}$ we have $\rho_{1,2}=+1$. Thus two periodic solutions $\Gamma_{1,1}^1$ and $\Gamma_{1,1}^2$ exist in the region $G_{1,1}$. The solution $\Gamma_{1,1}^2$ is unstable, the bifurcation diagram of the solution $\Gamma_{1,1}^1$ is symmetrical about the straight line $a=0$ and contains two domains of stability bounded by the lines $b=0$, $l_{1,1}$ and $d_{1,1}$, $d_{1,1}$.

The periodic solutions $\Gamma_{3,1}^1$ and $\Gamma_{3,1}^2$ generated by the solutions $x_1(t)={}^2/_3 t$ and $x_2(t)={}^2/_3 t+{}^1/_3\pi$ were investigated in the region $G_{3,1}\{|a|\leqslant 0.25, 0\leqslant b\leqslant 2.2\}$. The initial points $N^1_{3,1}(0, {}^2/_3)$ and $N^2_{3,1}(1/3\pi, {}^2/_3)$. correspond to these solutions when $a=b=0$. When a and b increase from zero, the abscissas of the points remain unchanged. Curves showing the dependence of the initial velocity $x^{\cdot}(0)$ of the solutions $\Gamma_{3,1}^1$ and $\Gamma_{3,1}^2$ (the solid and dashed line) are presented in Fig.3a.

A calculation of the multiplying factors of the solution $\Gamma_{3,1}^2$ shows its instability in the region $G_{3,1}$. The bifurcation diagram of the solution $\Gamma_{3,1}^1$ symmetrical about the straight line $a=0$, is shown in Fig.3b. The values of ρ_1 and ρ_2 in the region bounded by the lines $l_{3,1}$ and $d_{3,1}$ (vertical shading) are negative, and positive at the points (a, b) lying above the curve $s_{3,1}$ (oblique shading). On the curves $l_{3,1}$ and $d_{3,1}$ we have $\rho_{1,2}=-1$, and on $s_{3,1}$ we have $\rho_{1,2}=1$. The solution $\Gamma_{3,1}^1$ is stable in the non-shaded areas and here we have $\rho_{1,2}=\exp(\pm i\varphi(a, b))$.

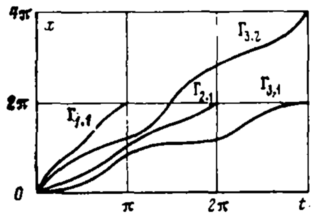

Fig.1

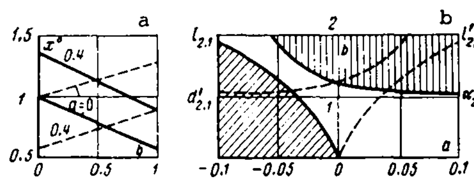

Fig.4

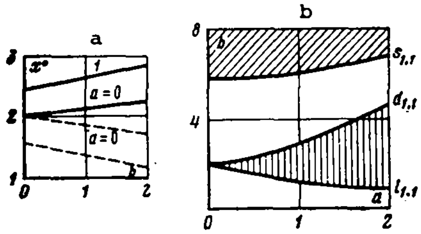

Fig.2

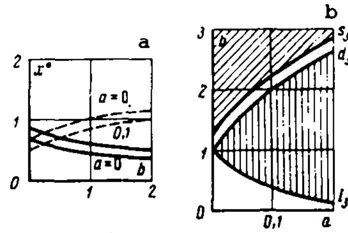

Fig.3

Analytical investigation and numerical study of other solutions $\Gamma_{p,q}$ carried out in Sect.2 enable us to formulate the following assertion for odd p and q.

Theorem 3. In the bounded domain of variation of the parameter a and b there exist, for any pair of prime p and q, two periodic solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$. The solution $\Gamma^2_{p,q}$ generated by the solution $x_2(t)=(2tq+\pi)/p$ is unstable. The bifurcation diagram of the solution $\Gamma^1_{p,q}$ generated by the solution $x_1(t)=2tq/p$ contains two regions of stability bounded by the lines $b=0$, $l_{p,q}$ and $d_{p,q}$, $s_{p,q}$. When the stability changes on the curves $l_{p,q}$ and $d_{p,q}$ the multiplying factors $\rho_{1,2}=-1$ on the curve $s_{p,q}$ the multiplying factors $\rho_{1,2}=+1$. For any a, b the abscissas of the initial points of the solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ are zero and π/p **respectively.**

Next we shall consider the periodic solutions $\Gamma_{p,q}$ with even p and q.

The solutions $\Gamma_{2,1}^i$ ($i=1, 2, 3, 4$) were studied in the region $G_{2,1}\{|a|\leqslant 0.45, 0\leqslant b\leqslant 1.6\}$. The solutions $\Gamma_{2,1}^1$ and $\Gamma_{2,1}^2$ are generated by the solutions $x_1(t)=t$ and $x_2(t)=t+{}^1/_2\pi$. A calculation of the coordinates of the initial points $N_{2,1}^1$ and $N_{2,1}^2$ showed that their abscissas are equal to zero and $\pi/2$ respectively for any a and b. The graphs in Fig.4a show the change in the initial velocity $x'(0)$ of the solutions $\Gamma_{2,1}^1$ and $\Gamma_{2,1}^2$ (the solid and dashed lines). Fig.4b shows the bifurcation diagram of the solution $\Gamma_{2,1}^1$ The solution $\Gamma_{2,1}^1$ is unstable within the shaded regions. At points of the region situated above the curve $d_{2,1}$ (vertical shading) the multiplying factors are negative, and in the region bounded by the half straight line $a<0, b=0$ and the curve $l_{2,1}$ they are positive. The non-shaded region represents the domain of stability of the solution $\Gamma_{2,1}^1$, and here $\rho_{1,2}=\exp(\pm i\varphi(a,b))$.

The bifurcation diagram of the solution $\Gamma_{2,1}^2$ is symmetrical with respect to the diagram of the solution $\Gamma_{2,1}^1$ about the b axis. The domain of stability of the solution $\Gamma_{2,1}^2$ is bounded by the lines $d_{2,1}', l_{2,1}'$ and the half straight line $a<0, b=0$. When the stability of the solution $\Gamma_{2,1}^2$ changes on the curve $l_{2,1}'$ two unstable solutions $\Gamma_{2,1}^3$ and $\Gamma_{2,1}^4$ emerge. When computing the coordinates of their initial points $N_{2,1}^3$ and $N_{2,1}^4$, we found that when the point (a, b) moves from the curve $l_{2,1}'$ to $l_{2,1}$, the points move away from the point $N_{2,1}^2$ and approach the point $N_{2,1}^1$. On the curve $l_{2,1}$ the points $N_{2,1}^3$ and $N_{2,1}^4$ merge with $N_{2,1}^1$. This causes a change in stability of the solution $\Gamma_{2,1}^1$. Thus the domain of existence of the solutions $\Gamma_{2,1}^3$ and $\Gamma_{2,1}^4$ is a set of points bounded by the curves $l_{2,1}$ and $l_{2,1}'$ emerging from the point $(0,0)$. A calculation of the multiplying factors of the solutions $\Gamma_{2,1}^3$ and $\Gamma_{2,1}^4$ showed their instability in this region.

Analogous results were obtained for the solutions $\Gamma^i_{3,2}$ $(i=1,2,3,4)$ in the region $G_{3,2}\{|a|\leqslant 0.8, 0\leqslant b\leqslant 1.2\}$ (Fig.5a). The stability of the solutions $\Gamma_{3,2}^1$ and $\Gamma_{3,2}^2$ changes on the curves $l_{3,2}$ and $l_{3,2}'$ (Fig.5b) respectively, and this leads to the appearance of two unstable solutions $\Gamma_{3,2}^3$ and $\Gamma_{3,2}^4$. In the ab plane the domain of stability of the solutions $\Gamma^1_{3,2}$ $(\Gamma^2_{3,2})$ is bounded by the curves $l_{3,2}$ and $d_{3,2}$ and the half straight line $a>0, b=0$ $(l_{3,2}', d_{3,2}'$ and $a<0, b=0)$.

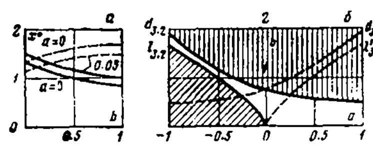

Fig.5

Analogous results were obtained when investigating a number of other solutions $\Gamma_{p,q}$ in the case when p and q were even. The region $G_{1,0}\{|a|\leqslant 1, 0\leqslant b\leqslant 10\}$ containing the first domain of stability and "beak- shaped" inclusions of the instability of stationary solutions $x(t)=0$ and $x(t)=\pi$ were considered for $q=0$ (see also the references shown in the second and third footnotes).

Theorem 4. When the numbers p and q are prime and one of them is even (including $q=0$), two periodic solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ generated by the solutions $x_1(t)=2tq/p$ and $x_2(t)=(2tq+\pi)/p$ **exist in the bounded domain of variation of the parameters a and b. The bifurcation** diagram of the solution $\Gamma^1_{p,q}$ $(\Gamma^2_{p,q})$ contains a single domain of stability bounded by the curves $d_{p,q}, l_{p,q}$ and half straight lines $a>0, b=0$ $(d_{p,q}', l_{p,q}'$ and $a<0, b=0)$. At the points of the curves $l_{p,q}$ and $l_{p,q}'$ and the straight line $b=0$, we have $\rho_{1,2}=+1$, and on the curves $d_{p,q}$ and $d_{p,q}'$ we have $\rho_{1,2}=-1$. Moreover, in the region bounded by the curves $l_{p,q}$ and $l_{p,q}'$ emerging from the point $(0,0)$, we have two unstable solutions $\Gamma^3_{p,q}$ and $\Gamma^4_{p,q}$ appearing when the solution $\Gamma^1_{p,q}$ $(\Gamma^2_{p,q})$ changes its stability on the curve $l_{p,q}$ $(l_{p,q}')$. The abscissas of the initial points of the solutions $\Gamma^1_{p,q}$ and $\Gamma^2_{p,q}$ are equal to zero and π/p respectively for any a and b. When the point (a, b) moves from the line $l_{p,q}$ towards $l_{p,q}'$ the abscissa of the initial point of the solution $\Gamma^3_{p,q}$ $(\Gamma^4_{p,q})$ varies from zero to π/p $(-\pi/p)$.

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